

THERMAL STABILITY OF A FLUID LAYER IN A VARIABLE GRAVITATIONAL FIELD IN POROUS MEDIUM

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for large scale convection phenomena occurring in the atmosphere, the ocean or the mantle of the earth. It then

ABSTRACT

The instability of a heated layer of viscous fluid in porous medium confined between two horizontal planes and subjected to variable gravitational field varying spatially with height is investigated. For a layer confined between two stress free boundaries, irrespective of whether gravitational acceleration is increasing or decreasing with height. It is shown that (i) the principal of exchange of stabilities is valid, when it is heated from below, & (ii) the layer is stable when it is heated from above. In the latter case, the complex growth rate of an arbitrary oscillatory mode lies out side of a circle whose radius depends on the wavelength of the mode and the Prandtl Number of the fluid but not on the Rayleigh Number. The underlying characterizing value problem is solved approximately for a linearly varying gravity and stress free boundaries. It is found that in the case of a layer heated from below, gravity increasing upward is a destabilizing influence, and also, we get increases of Rayleigh Number with the increasing of Permeability Constant. Critical Wave Number (a_c) decreases with the increases of gravity parameter in magnetic field at constant permeability constant.

Keywords: viscous fluid, Rayleigh Number, Critical Wave Number, Prandtl Number

1. INTRODUCTION

The Importance of porous medium and convection currents in our environment can scarcely be overestimated, we need only look to our immediate environment and note that the circulation of the Earth's atmosphere could not be explained without reference to convective motion in porous medium induced by solar heating. However, the idealization of uniform gravity assumed in the theoretical investigations, although valid for laboratory purpose, can scarcely be justified

becomes imperative to consider gravity as a variable quantity varying with distance from a reference point or surface.

Although Gresho and Sani³ have considered a time varying gravitational field in the convection problem of a horizontal fluid layer, the problem of a spatially varying gravitational field in the plane layer case still remains open. Pradhan and Patra considered the problem of onset of thermal instability in a cylindrical shell of self gravitating fluid heated internally and from below i.e. from the inside boundary. In case of Snell heated from below, the narrow gap approximation led to the plane layer problem with the gravity field varying linearly with height from the bottom surface. The prompted Pradhan and Samal⁵ to consider the plane layer problem under a spatially varying gravity field, neglecting viscosity. They found, among other things that if gravity remained downward (upward) throughout the flow domain, neutral modes did not exist. A sufficient condition for stability of a layer heated from above was that gravity remains directed downward over a sufficiently large part of the flow domain. However, a sufficient condition for instability of a layer heated from below was that, besides remaining directed downward, the gravity profile must have concave curvature throughout the flow domain. There also existed a circle limiting the growth rates of an arbitrary oscillatory mode of the system. These results, which are indicative of the behavior expected in the limit of small viscosity in the fluid, are here extended to the case of finite viscosity.

Present paper is the extension of the paper of G.K. Pradhan, P.C. Samal and U.K. Tripathy (1989) with the application of porous media.

2. PERTURBATION EQUATIONS

Consider a layer of an incompressible, viscous fluid in porous medium statically confined between two horizontal boundaries $z=0$ and $z=d$ which are maintained at constant temperature T_0 & T_1 respectively. The governing equations for flow and temperature, under the Boussinesq approximation (water waves), are

$$\rho_0 \left(\frac{\partial \bar{u}}{\partial t} + \bar{u} \cdot \nabla \bar{u} \right) = -\nabla p - \rho_0 g(z) \bar{k} [1 - \alpha(T - T_0)] + \mu \nabla^2 \bar{u} - \frac{\mu}{k_0} \bar{u} \quad \text{-----(1)}$$

$$\nabla \cdot \bar{u} = 0 \quad \text{.....(2)}$$

$$\left(\frac{\partial T}{\partial t} + \bar{u} \cdot \nabla T \right) = \kappa \nabla^2 T \quad \text{..... (3)}$$

. Where, ρ is the density of the fluid; $\bar{u}=(u, v, w)$ the fluid velocity; $g(z)$ the gravitational acceleration in a coordinate system $oxyz$ having oz , oy in the horizontal plane; $\bar{k}$ is a unit vector in the vertically upward direction; p is the pressure in the fluid; μ the dynamic viscosity; T the absolute temperature; κ is the thermal diffusivity; ρ_0 the density at temperature T_0 and K_0 be the permeability coefficient of porous medium

In the equilibrium state heat transport is by conduction alone and there are no velocities in the fluid. The solution of equation (1) to (3) is easily obtained in this case as the solution of (the basic state is denoted by an over bar)

$$\bar{u} = 0 \quad \text{... .. (4)}$$

$$\nabla \bar{p} = -g \rho_0 \bar{k} [1 - \alpha(\bar{T} - T_0)] \quad \text{.....(5)}$$

$$\nabla^2 \bar{T} = 0 \quad \text{..... (6)}$$

The basic temperature $\bar{T}$ is assumed to depend only on z , so we can integrate (6) and get $\bar{T} = T_0 + \beta z$ (7)

Where, $\beta = \frac{T_0 - T_1}{d}$ is negative, since this is the solution which gives us $T = T_0$ & T at $z = 0, d$ respectively.

$$\text{Writing, } p = \bar{p} + p'; T = \bar{T} + T'; \rho = \bar{\rho} + \rho' \text{ and } \bar{u} = \bar{u}' \quad \text{.....(8)}$$

Where, p', T', ρ' and $\bar{u}'$ are small perturbations with respect to which the equations are linearized and using the Boussinesq approximation, we obtain

$$\frac{\partial \bar{u}'}{\partial t} = -\frac{1}{\rho_0} \nabla p' + \nu \nabla^2 \bar{u}' + g \alpha T' \bar{k} - \frac{\nu}{K_0} \bar{u}' \quad \text{..... (9)}$$

$$\nabla \cdot \bar{u}' = 0 \quad \text{..... (10)}$$

$$\frac{\partial T'}{\partial t} + \beta w = \kappa \nabla^2 T' \quad \text{..... (11)}$$

Where, ν is the kinematic viscosity.

Seeking on x, y and t is given by

$$\exp i(k_x x + k_y y + \sigma t) \quad \text{..... (12)}$$

And eliminating p', u' and v' from the resulting equations. We have

$$\nu(D^2 - k^2) \left(D^2 - \frac{i\sigma}{\nu} + \frac{1}{K_0} - k^2 \right) w = k^2 g \alpha T' \quad \text{....(13)}$$

$$i\sigma T' - \kappa(D^2 - k^2)T' = -\beta w \quad \text{.... (14)}$$

Where, D denotes the differentiation with respect to z and $k^2 = k_x^2 + k_y^2$

Assuming the boundaries to be perfectly heat conducting and either rigid or free, we have

$T' = 0 = w$ for $z=0$ and d and either $Dw = 0$ (in case of rigid boundary)

$$\text{Or } D^2 w = 0 \text{ (in case of free boundary)} \quad \text{..... (15)}$$

Introducing the non-dimensional variables

$$z_* = \frac{z}{d}, \quad a = kd, \quad \sigma_* = \frac{i\sigma d^2}{\nu}, \quad w_* = \frac{wd}{\kappa}, \\ D_* = dD, \quad \theta_* = \frac{T'}{\beta d} \quad \text{..... (16)}$$

And $g(z) = g_0 \gamma(z_*)$ where, g_0 is the value of $g(z)$ at $z = 0$

We can write the equations (13) and (14), on suppressing the star subscript, as

$$\left(D^2 - a^2 - \sigma + \frac{d}{K_0} \right) (D^2 - a^2) w = -Ra^2 \gamma(z) \theta \quad \text{.... (17)}$$

$$(D^2 - a^2 - \sigma P_r) \theta = w \quad \text{..... (18)}$$

$$\text{Where, } R = \frac{-g_0 \alpha \beta d^4}{\nu \kappa} \quad \text{..... (19)}$$

and $P_r = \frac{\nu}{\kappa}$ (20)

are the non-dimensional Rayleigh and Prandtl Number respectively.

The boundary conditions reduce to $w = 0 = \theta$ ($z = 0, 1$) and either

$Dw = 0$ (at a rigid boundary)

Or $D^2w = 0$ (at a free boundary) (21)

Thus, for a given a , R and P_r , the equations (17) and (18) together with the boundary conditions (21) constitute an eigenvalues problem for σ and the system is unstable, neutral or stable according as the real part of σ , namely, σ_r is positive, zero or negative respectively.

3. THE PRINCIPAL OF EXCHANGE OF STABILITIES

For fixed stress free boundaries at $z = 0, 1$ we have the kinematic boundary condition as,

$W = D^2w = 0$ ($z = 0, 1$) (22)

Since the thermal boundary condition is

$\theta = 0$ ($z = 0, 1$) (23)

The boundary condition (22) can be expressed in terms of θ by means of (18) and (23) as

$D^2\theta = D^4\theta = 0$, ($z = 0, 1$) (24)

So finally $\theta = D^2\theta = D^4\theta = 0$, ($z = 0, 1$) (25)

Combining the equations (17) and (18), we have

$(D^2 - a^2 - P_r\sigma)(D^2 - a^2 - \sigma + \frac{1}{\kappa_0})(D^2 - a^2)\theta = -Ra^2\gamma(z)\theta$ (26)

which must be considered together with the boundary condition (25).

Multiplying eqn. (26) by θ^* , the complex conjugate of θ , and integrating the resulting equation over the range of z and making use the boundary conditions (25), we have

$P_r\sigma^2 \int_0^1 (|D\theta|^2 + a^2|\theta|^2) dz + \sigma(1 + P_r) \int_0^1 (D^2 - a^2)\theta|^2 dz - P_r\sigma \int_0^1 \frac{1}{\kappa_0} (|D\theta|^2 - a^2|\theta|^2) dz + \int_0^1 (|D^3\theta|^2 + 3a^2|D^2\theta|^2 + 3a^4|D\theta|^2 + a^6|\theta|^2) dz + \int_0^1 \frac{1}{\kappa_0} (D^2 - a^2)\theta|^2 dz + Ra^2 \int_0^1 \gamma(z)|\theta|^2 dz = 0$ (27)

Now, for a neutral mode, we must have $\sigma = i\sigma_i$ with σ_i real, and then the real and imaginary part of (27) gives us

$P_r\sigma_i^2 \int_0^1 (|D\theta|^2 + a^2|\theta|^2) dz - \int_0^1 (|D^3\theta|^2 + 3a^2|D^2\theta|^2 + 3a^4|D\theta|^2 + a^6|\theta|^2) dz - \int_0^1 \frac{1}{\kappa_0} (D^2 - a^2)\theta|^2 dz - Ra^2 \int_0^1 \gamma(z)|\theta|^2 dz = 0$ (28)

And,

$\sigma_i(1 + P_r) \int_0^1 (D^2 - a^2)\theta|^2 dz - P_r\sigma_i \int_0^1 \frac{1}{\kappa_0} (|D\theta|^2 - a^2|\theta|^2) dz = 0$ (29)

Eqn. (29) shows that $\sigma_i = 0$, then (28) always shows that the principle of exchange of stabilities is valid provided the neutral state exist, that is

$\int_0^1 (|D^3\theta|^2 + 3a^2|D^2\theta|^2 + 3a^4|D\theta|^2 + a^6|\theta|^2) dz - \int_0^1 \frac{1}{\kappa_0} (D^2 - a^2)\theta|^2 dz = Ra^2 \int_0^1 \gamma(z)|\theta|^2 dz$ (30)

Eqn. shows that, since $\gamma(z) > 0$, a necessary condition for the existence of neutral state is that

$R > 0$ (31)

If R is negative, there is no neutral state and a random disturbance will either be damped or amplified.

4. A SUFFICIENT CONDITION FOR STABILITY

Since $\sigma \neq 0$ for R negative in the case of stress-free boundaries, multiplying eqn. (27) throughout by σ^* , the complex conjugate of σ and dividing by $|\sigma|^2$, we have

$$P_r \sigma \int_0^1 (|D\theta|^2 + a^2 |\theta|^2) dz - (1 + P_r) \int_0^1 |(D^2 - a^2)\theta|^2 dz - \\ P_r \int_0^1 \frac{1}{K_0} (|D\theta|^2 - a^2 |\theta|^2) dz + \frac{\sigma^*}{|\sigma|^2} \int_0^1 (|D^3\theta|^2 + 3a^2 |D^2\theta|^2 + \\ 3a^4 |D\theta|^2 + a^6 |\theta|^2) dz \\ + \frac{\sigma^*}{|\sigma|^2} \int_0^1 \frac{1}{K_0} |(D^2 - a^2)\theta|^2 dz + Ra^2 \frac{\sigma^*}{|\sigma|^2} \int_0^1 \gamma(z) |\theta|^2 dz = 0$$

..... (32)

If $\sigma = \sigma_r + i\sigma_i$, then separating the real and imaginary part of eqn. (32) is

$$P_r \sigma_r \int_0^1 (|D\theta|^2 + a^2 |\theta|^2) dz + (1 + P_r) \int_0^1 |(D^2 - a^2)\theta|^2 dz - \\ P_r \int_0^1 \frac{1}{K_0} (|D\theta|^2 - a^2 |\theta|^2) dz + \frac{\sigma_r}{|\sigma|^2} \int_0^1 (|D^3\theta|^2 + 3a^2 |D^2\theta|^2 + \\ 3a^4 |D\theta|^2 + a^6 |\theta|^2) dz - Ra^2 \frac{\sigma_r}{|\sigma|^2} \int_0^1 \gamma(z) |\theta|^2 dz \\ = 0$$

.....(33)

And,

$$P_r \sigma_i \int_0^1 (|D\theta|^2 + a^2 |\theta|^2) dz - \frac{\sigma_i}{|\sigma|^2} \int_0^1 (|D^3\theta|^2 + 3a^2 |D^2\theta|^2 + \\ 3a^4 |D\theta|^2 + a^6 |\theta|^2) dz - \frac{\sigma_i}{|\sigma|^2} \int_0^1 \frac{1}{K_0} |(D^2 - a^2)\theta|^2 dz + \\ Ra^2 \frac{\sigma_i}{|\sigma|^2} \int_0^1 \gamma(z) |\theta|^2 dz = 0$$

..... (34)

Eqn. (34) shows that if $R < 0$, then $\sigma_r < 0$ provided $\gamma(z) > 0$ over most of the layer thus implies that stability for the configuration.

5. A CIRCULAR EXCLUSION THEOREM FOR OSCILLATORY MODES

Consider an arbitrary oscillatory mode of the system in the case of stress free boundaries. For such mode, we have $\sigma_i \neq 0$. Hence eqn. (34) can be put as

$$\int_0^1 (|D^3\theta|^2 + 3a^2 |D^2\theta|^2 + 3a^4 |D\theta|^2 + a^6 |\theta|^2) dz - \\ Ra^2 \int_0^1 \gamma(z) |\theta|^2 dz - P_r |\sigma|^2 \int_0^1 (|D\theta|^2 + a^2 |\theta|^2) dz + \\ \int_0^1 \frac{1}{K_0} |(D^2 - a^2)\theta|^2 dz = 0$$

..... (35)

This can be written as

$$\int_0^1 (|D^3\theta|^2 + 3a^2 |D^2\theta|^2 + 2a^4 |D\theta|^2) dz + \\ (a^4 - P_r |\sigma|^2) \int_0^1 (|D\theta|^2 + a^2 |\theta|^2) dz + \int_0^1 \frac{1}{K_0} |(D^2 - \\ a^2)\theta|^2 dz = Ra^2 \int_0^1 \gamma(z) |\theta|^2 dz$$

..... (36)

Hence, if $R < 0$, then we must have

$$\sigma_r^2 + \sigma_i^2 > \frac{a^4}{P_r} \quad \text{..... (37)}$$

Hence, the complex growth rate of an arbitrary oscillatory mode (in the case of a layer heated from above in a variable gravitational field directed downward) must lie outside of circle whose radius depends on the wavelength of the mode as well as the Prandtl Number of the fluid (but not on the Rayleigh number of the configuration).

6. THE PROBLEM OF A LAYER WITH FREE BOUNDARIES

We solve the characterizes value problem consisting of (17), (18) and (21) for free upper and lower boundaries for the case of a linearly varying gravity field

$$\gamma(z) = 1 + Mz > 0 \quad \text{..... (38)}$$

Since the principal of exchange of stabilities is true in this case and the marginal state is stationary, the equations to be solved are

$$(D^2 - a^2 + \frac{1}{K_0})(D^2 - a^2)w = -Ra^2(1 + Mz)\theta \quad \text{... (39)}$$

$$\text{And } (D^2 - a^2)\theta = w \quad \text{..... (40)}$$

Together with the boundary conditions

$$w = D^2w = \theta = 0, (z = 0, 1) \quad \text{..... (41)}$$

Eliminating w from eqns. (39) and (40), we get

$$(D^2 - a^2 + \frac{1}{K_0})(D^2 - a^2)^2\theta = -Ra^2(1 + Mz)\theta \quad \text{.... (42)}$$

The kinematic boundary condition can be expressed in terms of θ by eqn. (24). In view of (39) and (40), this gives

$$D^4\theta = D^6\theta = D^8\theta = \dots \dots \dots = D^{2n}\theta = 0, (z = 0, 1)$$

..... (43)

So θ must satisfy

$$\theta = D^2\theta = D^4\theta = \dots \dots \dots = D^{2n}\theta = 0, (z = 0, 1)$$

..... (44)

We have to solve eqn. (42) with the boundary condition (44). Rewrite eqn. (42) in the manner

$$\left(D^2 - a^2 + \frac{1}{K_0}\right)(D^2 - a^2)^2\theta = (1 + Mz)\varphi \quad \dots\dots (45)$$

$$\text{And } \varphi = -Ra^2\theta \quad \dots\dots\dots (46)$$

We expand φ and θ in the forms

$$\varphi = \sum_{m=1}^{\infty} A_m \sin m\pi z \text{ and } \theta = \sum_{m=1}^{\infty} -\frac{A_m \sin m\pi z}{Ra^2} \quad \dots (47)$$

Where, θ is the solution of the equation

$$\left(D^2 - a^2 + \frac{1}{K_0}\right)(D^2 - a^2)^2\theta = (1 + Mz) \sum_{m=1}^{\infty} A_m \sin m\pi z \quad \dots\dots\dots (48)$$

which satisfies the boundary conditions (44).

Inserting the expressions (47) in eqn. (46) and multiplying the resulting equation by $\sin n\pi z$ and integrating from 0 to 1 gives a set of homogeneous linear algebraic equation in the A_m . The requirement that the constant A_m in the resulting algebraic equation are not all zero, leads to the secular (characteristic) equation. An approximation to the solution can be obtained by setting the determinant formed by the element in the first n-rows and n-columns of the secular determinant equal to zero. Successive approximation will be obtained as n-increases. The calculation can be done for various values of M in each are the value of R which corresponds to a particular values of a being found by solving an algebraic equation. The minimum value of R with respect to 'a' can be found out for each M.

A first approximation to the solution of the secular equation is obtained by setting the (1, 1) – element of the matrix equal to zero and ignoring all the others. This corresponds to the choice of $\sin \pi z$ as a trial function for as Chandrasekhar^{1, 2} has shown very good approximation are obtained for $n = 1 = m$ and the solution converges rapidly for increasing values of m and n. Setting the (1, 1)-element to the secular determinant equal to zero, we find that

$$\frac{1}{2} \left(\pi^2 + a^2 - \frac{1}{K_0} \right) (\pi^2 + a^2)^2 \frac{1}{Ra^2} = \frac{1}{4} M + \frac{1}{2}$$

$$\text{Or } R = \frac{2}{2+M} \cdot \frac{(\pi^2 + a^2 - \frac{1}{K_0})(\pi^2 + a^2)^2}{a^2} \quad \dots\dots (49)$$

7.DISCUSSION

Numerical results summarized in fig. 1 to 3, obtained by evaluating eqn. (49). For comparison purpose, separates curves have been drawn using different values of permeability constant and gravity parameter. It is seen that a variable

gravitational field with increasing of gravity upward has a de-stability effect.

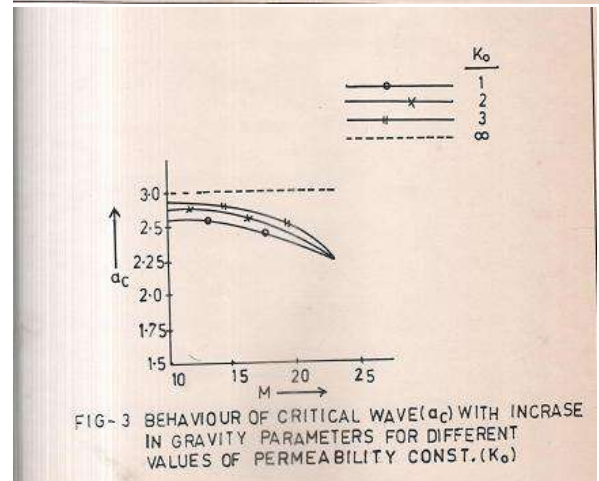
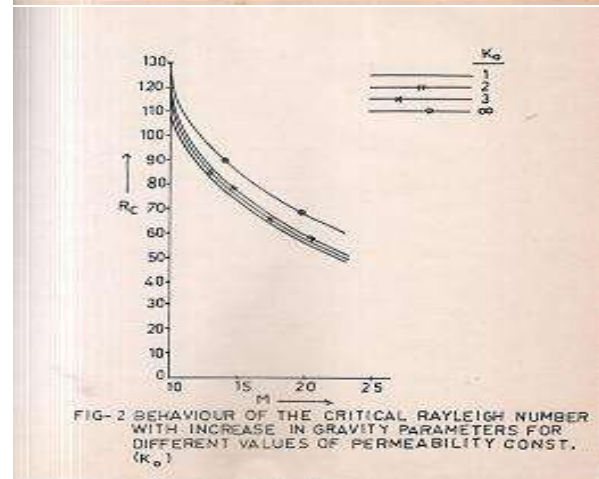
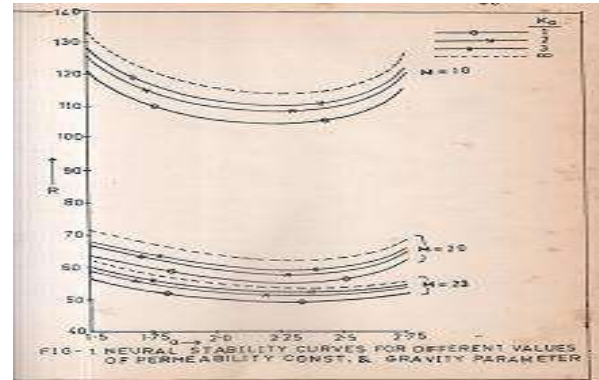


Fig.-1, shows the stability curves shifting progressively downward as the gravity parameter increases, but the stability curves shifting progressively upward as the permeability constant increases. The monotonic increase in the destabilizing affect with increase in gravity parameter, but permeability constant increase, the Rayleigh number is also increase in fig.-2. Fig.-3 shows that if permeability constant

increases, then critical Wave number is also increase as gravity parameter increase.

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